Experimental and theoretical study of the noise-induced gain degradation in vibrational resonance

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We present the results of theoretical and experimental investigations of the effect of additive noise on vibrational resonance in a bistable system driven by two periodic forces with very different frequencies. The phenomenon shows up as a parametric amplification of the low-frequency signal depending on the amplitude of high-frequency modulation. A scaling law for noise-induced decreasing of the gain factor, found theoretically, is compared with the experimental results obtained in a bistable vertical cavity surface emitting laser.

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Recently, considerable attention has been devoted both theoretically and experimentally to the phenomenon of vibrational resonance (VR), which appears in bistable systems being excited by two periodic signals with very different frequencies [1–8]. A distinctive feature of VR is that the response at the low frequency (LF) signal passes through a maximum depending on the amplitude of an additional high-frequency (HF) modulation. Such a method of the excitation has been proposed in the context of studying stochastic resonance (SR) in bistable systems where the added noise is replaced by the HF modulation [1]. An evidence of VR was demonstrated in analog electric circuits utilized to model noise-induced structures [3], excitable systems [4] and an overdamped bistable oscillator [5]. A theoretical explanation of the phenomenon VR was presented in [5,7] where it has been shown that VR takes place in the vicinity of the bifurcation corresponding to transition from bistability to monostability induced by the HF modulation. Recently, the experimental evidence of VR has been given in a vertical cavity surface emitting laser (VCSEL) operating in the regime of polarization bistability for both quasisymmetric and strongly asymmetric quasipotentials [6]. Besides, the possibility to make use of VR for low-level signal detection has been also demonstrated. It has been shown that the gain factor can reach very high values (about 300) for weak LF signals which significantly exceeds (by about 10 times) the gain factor due to SR obtained in the same conditions. From this point of view, the question of how noise affects the gain factor for very weak LF signals, is important from both theoretical and practical standpoints. Though, the effect of noise on VR was investigated numerically [5] and analytically [8], but no scaling law was found which relates the gain factor for a weak signal and the noise strength.

Here we present the results of theoretical and experimental investigations of the effect of additive noise on VR in VCSEL operating in the regime of the polarization bistability. Our theoretical consideration is based on the application of the effective potential to the theoretical results developed earlier for the study of SR. Due to the difference in time scales associated with LF and HF modulations, a rapidly oscillating double-well potential can be transformed into an effective potential with a parametric dependence on the amplitude and frequency of the HF modulation. Making use of such an approach, we find analytically a scaling law which relates the gain factor for the LF signal due to VR with the strength of the added noise, which is in a good agreement with the numerical and experimental results for weak LF signals.

Theoretically, the dynamics of the polarization switchings induced by noise and a deterministic modulation in the VCSEL can be described in the framework of a Langevin equation with a two-well potential [9–11]. Therefore, we consider the phenomenon of VR in the presence of noise using the model of an overdamped bistable oscillator. First, we consider the problem without noise. For sake of clarity, we reproduce here some of the results presented in [5,7]. The system is being excited by the LF signal $f(t)$ with a characteristic time τ_L and the amplitude A_L, where $f(t)$ can be a periodic or aperiodic function, and by a HF modulation with an amplitude A_H and a frequency Ω_H, such that $\Omega_H\tau_L \gg 1$. In this case, the dynamics is governed by the equation

$$\partial x/\partial t = -V'(x) + A_L f(t) + A_H \sin \Omega_H t, \tag{1}$$

where $V'(x)$ is the derivative with respect to x of a bistable potential function $V(x) = -\alpha x^2/2 + \beta x^4/4$, with local minima $x_m^{\pm} = \pm\sqrt{\alpha/\beta}$ and barrier height $\Delta V_0 = \alpha^2/4\beta$, where α and β are positive numbers. The dynamics is ruled by two time scales which are determined by LF and HF signals, respectively. Therefore we look for the solution as follows (the procedure is the same of Ref. [2]). We define $x(t) = y(t) + [A_H/(\Omega_H^2 + \eta^2)^{1/2}]\cos \Omega_H t$, where $y(t)$ denotes the slow part of the solution and η is a parameter. Substituting it into (1) and averaging over the period $T_H = 2\pi/\Omega_H$, we obtain the following equation which governs a slow dynamics of the system:

$$dY/dt = \alpha(1-\xi^2)Y - \beta Y^3 + A_L F(t), \tag{2}$$

where $Y = \langle y(t)\rangle_{T_H}$, $F(t) = \langle f(t)\rangle_{T_H}$, where $\langle z(t)\rangle_{T_H} = (1/T_H)\int_0^{T_H} z(T)dT$. We introduce here a normalized param-

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eter $\xi = A_H/\mu$, where μ is a switching threshold which depends on both the amplitude and the frequency

$$\mu = \sqrt{2\alpha\Omega_H^2/3\beta + 4\alpha^3/27\beta}. \quad (3)$$

In this case, the effective potential $V_{eff}(x)$ takes the form

$$V_{eff}(x,\xi) = -\alpha(1-\xi^2)x^2/2 + \beta x^4/4. \quad (4)$$

The sign near the quadratic term determines the character of $V_{eff}(x,\xi)$. For $\xi=1$ we have a bifurcation point where the transition from bistability to monostability occurs as the amplitude of the HF modulation increases [5,7] from which in the limit $\Omega_H \to 0$ the expression for μ (3) can be obtained. Obviously after averaging, the parameters of the potential such as the minima location $x_m^{\pm}$ and the barrier height ΔV depend on ξ

$$x_m^{\pm} = \pm\sqrt{\alpha(1-\xi^2)/\beta}, \quad \Delta V = \alpha^2(1-\xi^2)^2/4\beta. \quad (5)$$

Now, we can study the effect of noise on the system response to the LF signal $F(t)=\sin(\Omega_L t)$. In this case the equation reads

$$dY/dt = \alpha(1-\xi^2)Y - \beta Y^3 + A_L \sin(\Omega_L t) + \zeta(t), \quad (6)$$

where $\zeta(t)$ is a white, Gaussian noise with $\langle\zeta(t)\zeta(t')\rangle = 2D\delta(t-t')$ and mean $\langle\zeta(t)\rangle=0$. We explicitly assume here that the averaging does not change the character of noise when we add the noise term into the averaged equation. Equation (6) is the standard statement of the problem used for studying the phenomenon of SR. We can use therefore the analytical results obtained earlier, taking into account the dependence of the potential parameters on ξ. In particular, we consider here the well known result of the spectral amplification in SR. We define the gain factor G here as $G = R_L/R_0$, where R_L is the response of the system at the frequency Ω_L to an additional stochastic or HF modulation, whereas R_0 is the response in the linear approximation without any additional force. In this case $R_0 = A_L/\sqrt{\Omega_L + 4\alpha^2}$ or for $\Omega_L \ll 2\alpha$ the response R_0 reduces to $R_0 \cong A_L/2\alpha$. As shown in the limit $x_m A_L \ll D$, the response R_L can be evaluated from the following expression [see Eq. (3.7a) from [12]]:

$$R_L = \frac{A_L x_m^2}{D}\frac{2r_k}{(4r_k^2 + \Omega_L^2)^{1/2}}, \quad (7)$$

where

$$r_k = \frac{\alpha(1-\xi^2)}{\sqrt{2}\pi}\exp\left(-\frac{\alpha^2(1-\xi^2)^2}{4\beta D}\right) \quad (8)$$

is the Kramers rate. As x_m (5) and r_k (8) explicitly depend on ξ, we can analyze the effect of the additional HF modulation. Substituting x_m and r_k into Eq. (7), we obtain finally an expression for G_{VR}:

$$G_{VR} = \frac{2\sqrt{2}\alpha^3(1-\xi^2)^2\exp\left(-\frac{\alpha^2(1-\xi^2)^2}{4\beta D}\right)}{\beta D\sqrt{2\alpha^2(1-\xi^2)^2\exp\left(-\frac{\alpha^2(1-\xi^2)^2}{2\beta D}\right) + \Omega_L^2\pi^2}}. \quad (9)$$

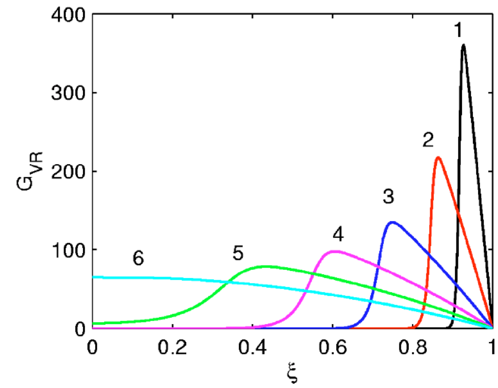

FIG. 1. (Color online) Analytics. Gain factor G_{VR} versus ξ for different values of the noise strength D=0.003 (1), 0.009 (2), 0.025 (3), 0.05 (4), 0.08 (5), and 0.12 (6) ($\Omega_L/2\pi$=0.0001).

Figure 1 shows G_{VR} as a function of ξ for different values of D. One can see the strong diminution of the gain factor G_{VR}, the broadening of the response curve and the shift of the maxima to the lower values of ξ as D increases. Obviously, the expression (9) can be used for the evaluation of G_{VR} only for $\xi<1$ since for $\xi \geq 1$ the bistable operation is lost. From the expression for G_{VR} (9), in the limit $\Omega_L \to 0$ one can find that the maximum G_{VR}^{max} approximately obeys the law

$$G_{VR}^{max} \cong 2\sqrt{2}\alpha\beta^{-1/2}\sqrt{\ln(\Omega_L^{-2}D)}D^{-1/2}. \quad (10)$$

In Fig. 2(a) we compare G_{VR}^{max} as a function of D plotted using Eq. (10) with values of G_{VR}^{max} obtained numerically from Eq. (9). One can note a good agreement between numerical and analytical results.

In order to check our predictions, we numerically integrated the following equation with the same meaning of all parameters as in analytical study:

$$dx/dt = 4(x - x^3) + A_L \sin\Omega_L t + A_H \sin\Omega_H t + \zeta(t). \quad (11)$$

In what follows, we use the normalized amplitudes ε and ξ defined as $\varepsilon = A_L/\mu_L$ and $\xi = A_H/\mu_H$ where μ_L and μ_H are the switching thresholds at the frequencies Ω_L and Ω_H, respectively. For a quantitative characterization of VR we used the gain factor G_{VR} defined as $G_{VR} = R_L(\Omega_L)/R_0(\Omega_L)$, where $R_L(\Omega_L)$ and $R_0(\Omega_L)$ are the responses at the low frequency in the presence and the absence of the HF modulation, respectively, which were evaluated from the spectra of the Fourier

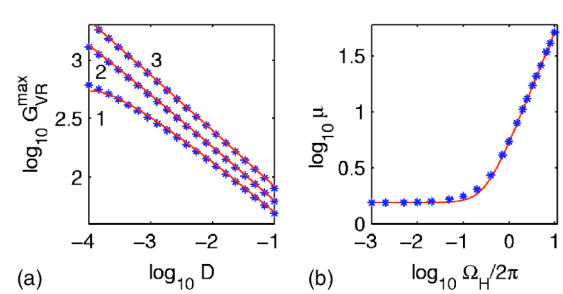

FIG. 2. (Color online) (a) G_{VR}^{max} vs D for different values of $\Omega_L = 10^{-3}$ (1), 10^{-4} (2), and 10^{-5} (3); (b) μ vs Ω_H (see text). Solid lines: analytics. Asterisks: numerics.

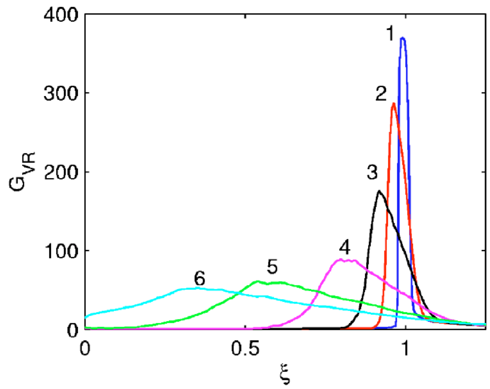

FIG. 3. (Color online) Numerics. The gain factor G_{VR} versus ξ for different values of the noise intensity $D=0.0001$ (1), 0.0026 (2), 0.011 (3), 0.048 (4), 0.15 (5), and 0.25 (6). $\Omega_L/2\pi=0.0001$; $\Omega_H/2\pi=0.1$: $\varepsilon=0.0162$.

transformed time series. A forward Euler algorithm with a fixed step of $0.001\times 2\pi/\Omega_H$ was used in the simulation.

First, in the absence of noise, we compare analytical approximation for μ (3) as a function of Ω_H with numerical results depicted in Fig. 2(b), showing a good agreement. Figure 3 shows the effect of noise on the gain factor G_{VR} for a weak LF signal ($\varepsilon=0.0162$). A strong degradation of G_{VR} as a noise strength D increases from $D=0.0001$ (curve 1) up to $D=0.25$ (curve 6) is observed. In particular, $G_{VR}^{\max}$ decreases by about 7 times in this case. The increase of the noise level leads to the broadening of the response curve and to the shift of the optimal value of ξ_{opt} corresponding to $G_{VR}^{\max}$. This picture is a good qualitative agreement with Fig. 1. For a certain value of D, which depends on ε, the effect of the excitation of VR completely disappears. This value of D corresponds to the optimal noise intensity D^* for the given amplitude ε in a conventional *SR* phenomenon [12]. This means that for all $D<D^*$ the gain factor for a weak LF signal due to VR will be larger than in conventional SR. For $D>D^*$ the additional HF modulation worsens G_{VR} for a weak LF signal.

In Fig. 4, $G_{VR}^{\max}$ as a function of D is shown for different values of ε. One can distinguish two regions in the figure. First, in a certain range of D, the value $G_{VR}^{\max}$ practically does not depend on D. This range decreases when lowering ε,

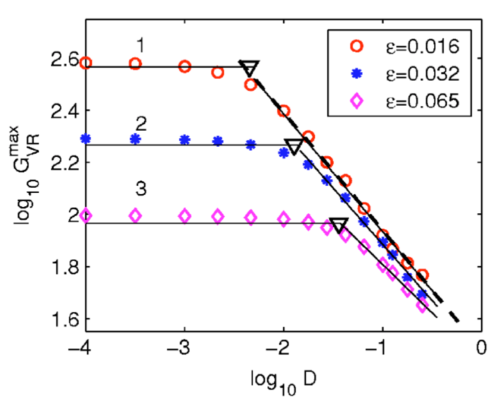

FIG. 4. (Color online) Numerics. $G_{VR}^{\max}$ versus D shown for different values of ε. A dashed line is plotted using Eq. (10). ($\Omega_L/2\pi=0.0001$; $\Omega_H/2\pi=0.01$.)

there, $G_{VR}^{\max}$ tends to a limiting value as D decreases. These limiting values can be evaluated from the simple relationship:

$$G_0 = 6\varepsilon^{-1}, \tag{12}$$

which can be obtained analytically in the limit of weak ε from Eq. (1) in the adiabatic regime of the two-frequency excitation in the absence of noise. In Fig. 4, the different G_0's are represented by horizontal lines. For a strong enough level of noise, the decreasing of $G_{VR}^{\max}$ in the log-log scale linearly depends on D (shown by inclined lines). A fitting of the numerical data relates $G_{VR}^{\max}$ and D as $G_{VR}^{\max}\sim D^{-\gamma}$, where γ depends on ε. The fit gives $\gamma\approx 0.46$, 0.45, 0.38 for $\varepsilon \cong 0.0162$, 0.0325, 0.065, respectively. One can note that the value of $\gamma\approx 0.46$ for a weak LF signal is in a good agreement with the analytical prediction [shown in Fig. 4 by a dashed line using Eq. (1)]. The critical value D_c, from which noise affects VR leading to its degradation, can be evaluated from the expression

$$D_c \cong (2/3)^{3/2}(\alpha^2/\beta)\varepsilon^{3/2}, \tag{13}$$

shown in Fig. 4 by triangles. This expression was obtained from the condition $x_m A_L = D$ in the same assumptions as Eq. (12). One can note that D_c approximately corresponds to the crossing points of horizontal and inclined lines in Fig. 4.

The experimental investigations performed in VCSEL have confirmed our main theoretical findings. The experimental setup was the same as in [6]. We studied VR in the laser intensity after polarization selection when two sinusoidal signals at the frequencies $\Omega_L=1$ kHz and $\Omega_H=100$ kHz with amplitudes A_L and A_H, respectively, were applied to the injection current. Both frequencies are much lower than the cutoff frequency of the laser polarization switching bandwidth. In what follows we use the normalized amplitudes of the LF and HF signals defined as $\varepsilon=A_L/\mu_L$ and $\xi=A_H/\mu_H$, where μ_L and μ_H are the switching thresholds at the frequencies Ω_L and Ω_H, respectively. The normalized amplitude of the HF signal ξ is the control parameter. Along with two periodic signals, noise with the different amplitudes σ_N was added to the injection current of the laser. In what follows we define the noise strength $D=\sigma_N^2$. The injection current was chosen in order that the laser operates in the regime of polarization bistability, where switching between two states could be induced by applying the deterministic modulation and noise. The laser responses were detected with a fast photodetector and recorded by a digital oscilloscope coupled with a computer to store and process the data. Every time series contained 5×10^4 sampling points with 20 periods of the LF signal. Each point of G_{VR} was obtained by averaging over 10 independent time series. As we already discussed, we expect that additive noise has a strong effect on VR leading to its degradation, when it added to the input signal, while $G_{VR}^{\max}$ obeys a simple scaling law. We present the results of an experimental evidence of this scaling law. The experimentally measured gain factor G_{VR} for a weak LF signal ($\varepsilon=0.028$) versus ξ is shown in Fig. 5 for different values of the added noise intensity. One can see that an addition of noise results in a diminution of the gain, the shift of its maximum and broadening the response curve

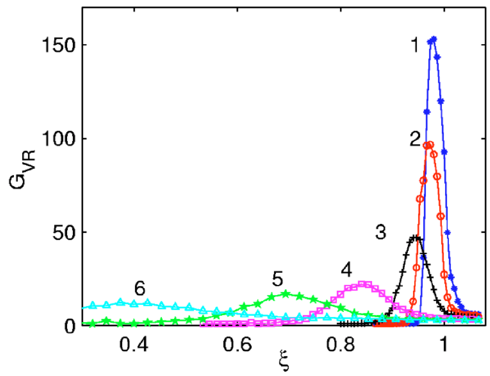

FIG. 5. (Color online) Experiment. The gain factor G versus the normalized amplitude ξ for different values of the noise level σ_N =7 (1), 14 (2), 30 (3), 65 (4), 110 (5), and 250 (6) (mV_{rms}). (ε =0.028.)

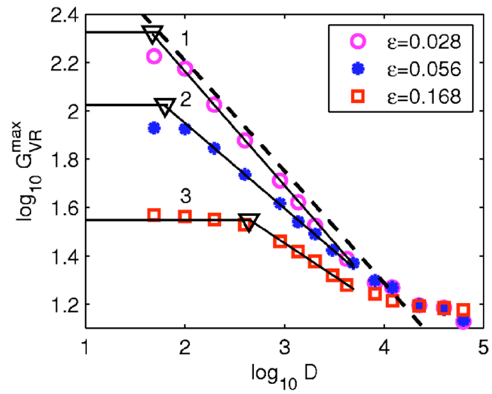

FIG. 6. (Color online) Experiment. $G_{VR}^{\max}$ versus D for different values of ε. A dashed line is plotted using Eq. (10) with rescaled D.

as a noise intensity D increases. These experimentally observed features are in a good qualitative agreement with the numerical and analytical results shown in Fig. 3 and Fig. 1, respectively. The experimentally measured $G_{VR}^{\max}$ versus D for three different values of the amplitude of the LF signal is shown in Fig. 6. In agreement with the results of the numerical simulation presented in Fig. 4, one can note that there exists a range of the noise intensity where $G_{VR}^{\max}$ practically does not depend on D (Fig. 6, curve 3). These values of $G_{VR}^{\max}$ are also in good agreement with the theoretical prediction (12) shown by horizontal lines. In the region of D where $G_{VR}^{\max}$ quickly decreases as D increases, the fitting of the experimental data yields the scaling $G_{VR}^{\max} \sim D^{-\gamma}$, where $\gamma \approx 0.47$, 0.34, 0.27 for $\varepsilon=0.028$, 0.056, 0.168, respectively. All values of γ agree with the results of the numerical simulation. Moreover, the first value of γ is in a good agreement with the value predicted analytically and shown by a dashed line in Fig. 6. We can therefore conclude that the experimental results confirm theoretical predictions concerning the behavior of $G_{VR}^{\max}$ depending on the LF amplitude and the noise strength.

To conclude, we have shown that the introduction of the effective potential into the theoretical results developed for the description of SR allows to study the effect of the additive noise on VR. We found analytically the scaling law relating the gain factor and the noise strength which is in an agreement with the numerical and experimental results. Our results are very general and one can expect that they can be applicable to bistable systems from different fields. On the other hand, one can expect also that this scaling law can be observed in a broad class of nonautonomous systems displaying period-doubling bifurcations, since the normal form equation describing the essential dynamics on a low-dimensional manifold coincides with the equation for an overdamped oscillator with parametrically dependent potential [13].

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